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Convergence from Boltzmann to Landau Processes with Soft Potential and Particle Approximations

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Our aim in this paper is to show how a probabilistic interpretation of the Boltzmann and Landau equations gives a microscopic understanding of these equations. We firstly associate stochastic jump processes with the Boltzmann equations we consider. Then we renormalize these equations following asymptotics which make prevail the grazing collisions, and prove the convergence of the associated Boltzmann jump processes to a diffusion process related to the Landau equation. The convergence is pathwise and also implies a convergence at the level of the partial differential equations. The best feature of this approach is the microscopic understanding of the transition between the Boltzmann and the Landau equations, by an accumulation of very small jumps. We deduce from this interpretation an approximation result for a solution of the Landau equation via colliding stochastic particle systems. This result leads to a Monte-Carlo algorithm for the simulation of solutions by a conservative particle method which enables to observe the transition from Boltzmann to Landau equations. Numerical results are given.

KEY WORDS: Soft potential Boltzmann equations without cutoff; Landau equation with soft potential; nonlinear stochastic differential equations; interacting particle systems; Monte-Carlo algorithm.

1. INTRODUCTION

Our aim in this paper is to show how a probabilistic interpretation of the Boltzmann and Landau equations gives a microscopic understanding of these equations.

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In the first part of the paper, we consider spatially homogeneous soft potential Boltzmann equations without angular cutoff for a large class of initial data, and relate them to jump processes solutions of Poisson-driven stochastic differential equations. These results extend results due to Tanaka in the Maxwellian case and for L^1-hypotheses on the cross-section[25] and generalized by Horowitz and Karandikar[18] in the L^2-case and by Fournier and Méléard[11] for non Maxwell molecules in dimension 2.

This probabilistic representation has been proved usefull either to obtain existence of measure solutions of the Boltzmann equation for a large class of measure initial data or to prove, at least in dimension two, the existence of positive smooth solutions to the Boltzmann equation, improving thus the analytical results (Graham and Méléard,[15] Fournier[9]). It also allows to get numerical Monte-Carlo methods for Boltzmann equations without cutoff (Desvillettes *et al.*,[7] Fournier and Méléard[12]).

In this paper we show more specifically that the microscopic stochastic representation of the Boltzmann equations leads to a natural and intuitive understanding of the transition to Landau equations, when grazing collisions prevail.

The Fokker–Planck–Landau equation, or Landau equation, is derived from the Boltzmann (see ref. 20) and is usually considered as an approximation of homogeneous Boltzmann equations in the limit of grazing collisions. Many authors have been interested in proving rigorously this convergence, in different cases of scaterring cross-sections and initial data. Firstly Arsen'ev and Buryak[1] proved the convergence of solutions of the Bolzmann equation towards solutions of the Landau equation under very restrictive assumptions. Then, Desvillettes[5] gave a mathematical framework for more physical situations, but excluding the main case of Coulomb potential studied by Degond and Lucquin,[4] for which the Boltzmann equation is not realistic (see ref. 27) and the Landau equation appears naturally. More recently, Goudon[13] and Villani[27] proved the convergence of Boltzmann equations towards the Landau equation. They use analytical techniques as convergence theorems or spectral analysis, showing a L^1-convergence for a bounded entropy and energy initial condition. However, these results could be relaxed without the entropy assumption in a weak-* convergence.

In the grazing collision asymptotics, the cross-section in the Boltzmann operator is renormalized by a small parameter depending on the nature of the collisions. In this paper, we consider asymptotics including those of Degond and Lucquin-Desreux[4] and Desvillettes.[5] We show how the accumulation of grazing collisions can be interpreted at the level of the jump processes as an accumulation of small jumps. Then we prove the convergence in law, in the Skorohod space, of sequences of renormalized

Boltzmann processes to a diffusion process, called Landau process, which describes the microscopic random behaviour of the Fokker–Planck–Landau equation (see ref. 16). We immediately deduce a convergence result at the level of the partial differential equations for general initial data. Unhappily, the probabilistic tools oblige us to use a L^2-framework, which necessitates the consideration of potentials $\gamma \in (-1, 0]$. In particular, our theorical approach does not recover the interesting Coulombian case, even if the Monte-Carlo algorithm also makes sense in this case.

As in the analytical framework, uniqueness is an open problem for all the equations we consider. All the convergence results we prove are obtained by a compactness method which only gives converging subsequences.

The pathwise interpretation of the equations (in the probabilistic framework) provides a natural approximation by interacting colliding particle systems of the Fokker–Planck–Landau equations. The collision rate and the amplitude of jumps of the particles are related to the size of the system. We prove the convergence of its empirical measures to a weak solution of the Landau equation, when the size of the system growths. We deduce from this theorical result a simple simulation algorithm, based upon particles conserving momentum and kinetic energy.

We finally discuss about numerical results. The main interest of our approach is to observe in the simulations the transition from the renormalized Boltzmann equations to the Landau equation (see Section 6, Fig. 1).

The paper is organized as follows: in Section 2, we explain the pathwise interpretation of the Boltzmann equation with soft potential, and solve the nonlinear Poisson-driven stochastic differential equation. In Section 3, we study the convergence in law of the renormalized Boltzmann processes to a Landau process and deduce the convergence of solutions of the Boltzmann equations to the ones of the Landau equation when the grazing collisions prevail. In Section 4, we study the approximating particle systems. We describe the pathwise Monte-Carlo algorithm in Section 5. Numerical results are discussed in Section 6.

Notations

– $\mathbb{D}_T$ will denote the Skorohod space $\mathbb{D}([0, T], \mathbb{R}^3)$ of càdlàg functions from $[0, T]$ into $\mathbb{R}^3$. The space $\mathbb{D}_T$ endowed with the Skorohod topology is a Polish space.

– $\mathscr{C}_T$ is the space $C([0, T], \mathbb{R}^3)$ of continuous functions from $[0, T]$ into $\mathbb{R}^3$ and $C_b^2(\mathbb{R}^3)$ is the space of real bounded functions of class $\mathscr{C}^2$ with bounded derivatives.

– $\mathscr{P}(\mathbb{R}^3)$ is the set of probability measures on $\mathbb{R}^3$ and $\mathscr{P}_2(\mathbb{R}^3)$ the subset of probability measures with a finite second order moment. Similarly, $\mathscr{P}(\mathbb{D}_T)$ denotes the space of probability measures on $\mathbb{D}_T$ and $\mathscr{P}_2(\mathbb{D}_T)$ is the subset of probability measures with a finite second order moment: $q \in \mathscr{P}_2(\mathbb{D}_T)$ if $\int_{x \in \mathbb{D}_T} \sup_{t \in [0,T]} |x(t)|^2 \, q(dx) < \infty$.

– Let A and B be two matrices with same dimensions. The symbol $A:B$ denotes the real $\sum_{i,j} A_{ij} B_{ij}$ and A^t is the transpose matrix of the matrix A.

– K will denote a real positive constant of which the value may change from line to line.

2. THE BOLTZMANN PROCESS

2.1. The Equation

The Boltzmann equation we consider describes the evolution of the density $f(t, v)$ of particles with velocity $v \in \mathbb{R}^3$ at time t in a rarefied homogeneous gas:

$$\frac{\partial f}{\partial t} = Q_B(f, f), \tag{2.1}$$

where Q_B is a quadratic collision kernel preserving momentum and kinetic energy,

$$\begin{aligned} Q_B(f, f)(t, v) = \int_{v_* \in \mathbb{R}^3} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} & (f(t, v') \, f(t, v'_*) - f(t, v) \, f(t, v_*)) \\ & \times B(|v - v_*|, \theta) \, d\theta \, d\varphi \, dv_* \end{aligned} \tag{2.2}$$

with $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2} \sigma$ and $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2} \sigma$, the unit vector σ having colatitude θ and longitude φ in the spherical coordinates in which $v - v_*$ is the polar axis. The nonnegative function B is called the cross-section.

We are interested in cases for which the molecules in the gas interact according to an inverse power law in $1/r^s$ with $s \geqslant 2$. The physical cross-sections $B(z, \theta)$ tend to infinity when θ goes to zero, but satisfy $\int_0^\pi |\theta|^2 \, B(z, \theta) \, d\theta < \infty$ for each z. Physically, this explosion near 0 comes from the accumulation of grazing collisions.

In this general (spatially homogeneous) setting, the Boltzmann equation is difficult to study. A large literature deals with the non physical equation with angular cutoff, namely under the assumption $\int_0^\pi B(z, \theta) \, d\theta < \infty$. More recently, the case of Maxwell molecules, for which the cross-section $B(z, \theta) = \beta(\theta)$ only depends on θ, has been studied without the cut-off assumption. In the Maxwell context, Tanaka[(25)] was considering the case where $\int_0^\pi \theta\beta(\theta)\, d\theta < \infty$, and Horowitz and Karandikar,[18] Desvillettes,[6] Fournier,[9] Fournier and Méléard,[12] have worked under the physical assumption $\int_0^\pi \theta^2\beta(\theta)\, d\theta < +\infty$. In the non Maxwell case, by analytical methods, Goudon[13] and Villani[27] obtain existence results. With a probabilistic approach, Fournier and Méléard[11] obtain such results in dimension 2 and for cross-sections bounded as velocity functions. We generalize here this approach in dimension 3 and for unbounded (as velocity field) soft potential cross-sections of the form

$$B(z, \theta) = \psi(z)\, \beta(\theta), \tag{2.3}$$

with

$$\psi(z) = h(|z|)\, |z|^\gamma,$$

$\gamma \in (-1, 0]$ and h a bounded nonnegative locally Lipschitz continuous function and β from $(0, \pi] \to \mathbb{R}+$ such that $\int_0^\pi \theta^2\beta(\theta)\, d\theta < \infty$.

Remark 2.1. The probabilistic tools oblige us to work in a L^2-framework and we are able to deal with moderately soft potentials, $\gamma \in (-1, 0]$, thanks to the usefull estimate: for each $\gamma \in (-1, 0]$, for each $z \in \mathbb{R}^3$,

$$|z|^{2+\gamma} \leqslant |z|^2 + 1; \qquad |z|^{2+2\gamma} \leqslant |z|^2 + 1. \tag{2.4}$$

We define the jump amplitude

$$a(v, v_*, \theta, \varphi) = v' - v = \frac{\cos\theta - 1}{2}(v - v_*) + \frac{\sin\theta}{2}\Gamma(v - v_*, \varphi). \tag{2.5}$$

where for $x \in \mathbb{R}^3$, $\varphi \in [0, 2\pi)$,

$$\Gamma(x, \varphi) = \cos\varphi I(x) + \sin\varphi J(x) \tag{2.6}$$

and $\frac{1}{|x|}(x, I(x), J(x))$ is an orthonormal basis of $\mathbb{R}^3$. One can choose, for example,

$$I(x) = \begin{cases} \dfrac{|x|}{\sqrt{x_1^2 + x_2^2}}(-x_2, x_1, 0) & \text{if } \ x_1^2 + x_2^2 > 0 \\ (x_3, 0, 0) & \text{if } \ x_1^2 + x_2^2 = 0 \end{cases}; \qquad J(x) = \frac{x}{|x|} \wedge I(x)$$

The main difficulty is that a is not a Lipschitz continuous function on the variables v and v_*. It just satisfies an "almost"-Lipschitz property

(Lipschitz up to a rotation), as proved in ref. 25 or in its "fine" version in ref. 12. However, this property will be sufficient to obtain existence results.

Lemma 2.2. There exists a measurable function $\varphi_0: \mathbb{R}^3 \times \mathbb{R}^3 \mapsto [0, 2\pi[$, such that for all v, v_*, w, w_* in $\mathbb{R}^3$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$,

$$|a(v, v_*, \theta, \varphi) - a(w, w_*, \theta, \varphi + \varphi_0(v - v_*, w - w_*))| \leqslant 3\theta(|v-w| + |v_* - w_*|)$$

$$|a(v, v_*, \theta, \varphi)| \leqslant 2\, |\sin(\theta/2)|\, |v - v_*|$$

Equation (2.1) has to be understood in a weak sense, i.e., f is a solution of the equation if for any test function ϕ, $\frac{\partial}{\partial t}\langle f, \phi\rangle = \langle Q_B(f, f), \phi\rangle$ where $\langle \cdot, \cdot\rangle$ denotes the duality bracket between L^1 and L^∞ functions. By a standard integration by parts, we define a solution f as satisfying for each $\phi \in C_b^2(\mathbb{R}^3)$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f(t, v)\, \phi(v)\, dv = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi} (\phi(v') - \phi(v))\, B(v - v_*, \theta)\, d\theta\, d\varphi \\ \times f(t, v)\, dv\, f(t, v_*)\, dv_*.$$

Since $\int_0^\pi \theta\beta(\theta)\, d\theta$ may be infinite, the RHS term may explode. Thus we have to compensate it, and taking into account the conservation of the mass, we obtain finally the following definition of probability measure solutions of (2.1).

Definition 2.3. We say that a probability measure family $(P_t)_{t \geqslant 0}$ is a measure-solution of the Boltzmann equation (2.1) if for each $\phi \in C_b^2(\mathbb{R}^3)$

$$\langle \phi, P_t\rangle = \langle \phi, P_0\rangle + \int_0^t \langle K^{\phi}_{\beta,\gamma}(v, v_*), P_s(dv)\, P_s(dv_*)\rangle\, ds, \tag{2.7}$$

where $K^{\phi}_{\beta,\gamma}$ is defined in the compensated form

$$K^{\phi}_{\beta,\gamma}(v, v_*) = -b\psi(v - v_*)(v - v_*) \cdot \nabla\phi(v) \\ + \int_0^{2\pi} \int_0^{\pi} (\phi(v + a(v, v_*, \theta, \varphi)) - \phi(v) - a(v, v_*, \theta, \varphi) \cdot \nabla\phi(v)) \\ \times \psi(v - v_*)\, \beta(\theta)\, d\theta\, d\varphi \tag{2.8}$$

and where

$$b = \pi \int_0^{\pi} (1 - \cos\theta)\, \beta(\theta)\, d\theta. \tag{2.9}$$

2.2. The Probabilistic Approach

We consider (2.7) as the evolution equation for the marginals of a Markov process which law is defined by a martingale problem.

Definition 2.4. Let β be a cross section such that $\int_0^\pi \theta^2 \beta(\theta)\, d\theta < +\infty$ and Q_0 in $\mathscr{P}_2(\mathbb{R}^3)$.

We say that $Q \in \mathscr{P}(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3))$ solves the nonlinear martingale problem (BMP) starting at Q_0 if under Q, the canonical process V satisfies for any $\phi \in C_b^2(\mathbb{R}^3)$

$$\phi(V_t) - \phi(V_0) - \int_0^t \int_{\mathbb{R}^3} K^{\phi}_{\beta, \gamma}(V_s, v_*)\, Q_s(dv_*)\, ds \tag{2.10}$$

is a square-integrable martingale and the law of V_0 is Q_0. Here, the nonlinearity appears through Q_s which denotes the marginal of Q at time s.

Remark 2.5. Taking expectations in (2.10), we remark that if Q is a solution of (BMP), then its time-marginal family $(Q_t)_{t \geqslant 0}$ is a measure-solution of the Boltzmann equation, in the sense of Definition 2.3.

Our first aim is to prove the existence of a solution to the martingale problem (2.10) and then to obtain the existence of a measure-solution to the Boltzmann equation.

Theorem 2.6. Assume that Q_0 is a probability measure on $\mathbb{R}^3$ with a fourth order moment, and that $B(z, \theta) = \psi(z)\, \beta(\theta)$ is a cross-section satisfying Hypothesis (2.3). Then

(1) The nonlinear martingale problem (BMP) with initial data Q_0 has a solution $Q \in \mathscr{P}_2(\mathbb{D}_T)$.

(2) Moreover, $E_Q(\sup_{t \leqslant T} |X_t|^4) < +\infty$, where X is the canonical process on $\mathbb{D}_T$.

Remark 2.7. There is no assumption on Q_0, except the existence of a fourth order moment. This allows us in particular to consider degenerate initial data, as Dirac measures. The point (1) in Theorem 2.6 exhibits in particular a measure-solution to the Boltzmann equation (2.1) for each initial data $Q_0 \in \mathscr{P}_4(\mathbb{R}^3)$.

Our method gives no hope to obtain a uniqueness result.

We will prove this theorem using stochastic calculus tools. We generalize here the results of Tanaka and Horowitz–Karandikar[18] to soft potential cases, introducing a specific nonlinear stochastic differential equation giving a pathwise version of the probabilistic interpretation.

We are looking for a stochastic process belonging to $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3)$ and with a law Q solution of (2.10). It can be given as solution of the nonlinear stochastic differential equation

$$V_t = V_0 - b\int_0^t \int_{\mathbb{R}^3} \psi(V_s - z)(V_s - z)\, Q_s(dz)\, ds$$

$$+ \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_0^{\pi} \int_0^{2\pi} a(V_{s-}, z, \theta, \varphi)\, \mathbf{1}_{\{x \leqslant \psi(V_{s-} - z)\}} \tilde{N}^*(dx, d\theta, d\varphi, dz, ds)$$

where $\tilde{N}^*$ the compensated martingale of an inhomogeneous Poisson-point measure on $\mathbb{R}_+ \times [0, \pi] \times [0, 2\pi] \times \mathbb{R}^3 \times \mathbb{R}_+$ with intensity $dx\, \beta(\theta)\, d\theta\, d\varphi \times Q_t(dz)\, dt$. The nonlinearity appears through Q_s, which is the law of V_s for each s.

We consider a compensated form of the Poisson-point measure following Definition 2.3. Using Itô's formula, we easily remark that the law Q of a solution V of this stochastic differential equation is a solution of (2.10) and $(Q_t)_{t \geqslant 0}$ is a solution of the Boltzmann equation. That gives a pathwise mean-field interacting representation of the Boltzmann process: the process jumps following a Poisson-point measure which picks independent colliding particules having the same law as the process itself. The jump takes place if $x \leqslant \psi(V_{s-} - z)$ and the amplitude of the jump is equal to a.

Technically, to obtain a more intrinsic representation, we use the Skorohod representation and describe the behaviour of the colliding particules on an auxiliary probability space. So we now consider two probability spaces: the first one is the abstract space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \in [0,T]}, P)$ and the second one is $([0, 1], \mathscr{B}([0, 1]), d\alpha)$. In order to avoid any confusion, the processes on $([0, 1], \mathscr{B}([0, 1]), d\alpha)$ will be called α-processes, the expectation under $d\alpha$ will be denoted by E_α, and the laws $\mathscr{L}_\alpha$.

Definition 2.8. We say that (V, W, N, V_0) is a solution of (SDE) if

(i) (V_t) is an adapted càdlàg $\mathbb{D}_T$-valued process such that $E(\sup_{t \in [0,T]} |V_t|^2) < +\infty$,

(ii) (W_t) is a α-process such that $E_\alpha(\sup_{t \in [0,T]} |W_t|^2) < +\infty$,

(iii) $N(\omega, dt, d\alpha, dx, d\theta, d\varphi)$ is a $\{\mathscr{F}_t\}$-Poisson point measure on $[0, T] \times [0, 1] \times \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi]$ with intensity $m(dt, d\alpha, dx, d\theta, d\varphi) = dt\, d\alpha\, dx\, \beta(\theta)\, d\theta\, d\varphi$ and $\tilde{N}$ is its compensated martingale,

(iv) V_0 is a square integrable variable independent of N,

(v) $\mathscr{L}(V) = \mathscr{L}_\alpha(W)$,

(vi)

$$V_t = V_0 - b \int_0^t \int_0^1 \psi(V_s - W_s(\alpha))(V_s - W_s(\alpha))\, d\alpha\, ds$$

$$+ \int_0^t \int_0^1 \int_{\mathbb{R}_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}, W_{s-}(\alpha), \theta, \varphi)\, \mathbf{1}_{\{x \leqslant \psi(V_{s-} - W_{s-}(\alpha))\}}$$

$$\times \tilde{N}(ds, d\alpha, dx, d\theta, d\varphi)$$

Remark 2.9. Of course, as before, the law of V is then a solution of (BMP) with initial law $Q_0 = \mathscr{L}(V_0)$.

Let us now prove in many steps Theorem 2.6. We obtain the existence of weak solutions of the martingale problem (BMP) under Hypothesis (2.3), as limits in law of solutions of regularized equations.

The first step generalizes the result of Fournier and Méléard[11] obtained in dimension 2. The specific difficulty in dimension 3 is the lack of Lipschitz continuity of a described in Lemma 2.2. We will prove

Proposition 2.10. Assume that $B(z, \theta) = \hat{\psi}(z)\, \beta(\theta)$ with $\hat{\psi}$ a non-negative bounded and locally Lipschitz continuous function, and β integrating θ (hence, no compensation is needed). Assume that V_0 is a fourth-order moment random variable. Then the nonlinear stochastic differential equation (SDE) which can be rewritten in this case

$$V_t = V_0 + \int_0^t \int_0^1 \int_{\mathbb{R}_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}, W_{s-}(\alpha), \theta, \varphi)$$

$$\times \mathbf{1}_{\{x \leqslant \hat{\psi}(V_{s-} - W_{s-}(\alpha))\}} N(ds, d\alpha, dx, d\theta, d\varphi) \tag{2.11}$$

has a weak solution, and moreover, for every $T > 0$,

$$E(\sup_{t \leqslant T} |V_t|^4) < +\infty. \tag{2.12}$$

Proof. The proof mixes arguments from ref. 12 (adapted from Tanaka) to control the lack of Lipschitz regularity of a and from ref. 11 Theorem 3.4. in the non Maxwell case.

Let us assume that the function $\hat{\psi}$ is bounded by M. Let us define

$$\hat{a}(v, w, \theta, \varphi, x) = a(v, w, \theta, \varphi)\, \mathbf{1}_{\{x \leqslant \hat{\psi}(v-w)\}}$$

and its cutoff versions

$$\hat{a}_n(v, w, \theta, \varphi, x) = \hat{a}(v \wedge n \vee (-n), w \wedge n \vee (-n), \theta, \varphi, x).$$

We remark that

$$\int |\hat{a}_n(v, w, \theta, \varphi, x)| \, dx \leqslant M\theta \, |v-w| \tag{2.13}$$

$$\int |\hat{a}_n(v, w, \theta, \varphi, x) - \hat{a}_n(v', w', \theta, \varphi + \varphi_0(v-w, v'-w'), x)| \, dx$$

$$\leqslant K_n(|v-v'| + |w-w'|) \tag{2.14}$$

Thanks to these properties, we are able to construct, by a sophisticated Picard iteration mixing results of refs. 11 and 12, a solution of

$$V_t^n = V_0 + \int_0^t \int_0^1 \int_{\mathbb{R}_+} \int_0^\pi \int_0^{2\pi} \hat{a}_n(V_{s-}^n, W_{s-}^n(\alpha), \theta, \varphi, x) \, N(ds, d\alpha, dx, d\theta, d\varphi) \tag{2.15}$$

satisfying moreover that

$$\sup_n E(\sup_{s \leqslant t} |V_s^n|^4) < +\infty. \tag{2.16}$$

This Picard iteration takes into account the specific property (2.14). The trick is to observe that the image measure of a Poisson point measure with intensity $ds \, dx \, d\alpha \, \beta(\theta) \, d\theta \, d\varphi$ by the rotation $\varphi \mapsto \varphi + \varphi_0$ is still a Poisson point measure with the same intensity measure. That is technical and we refer to refs. 12 or 25 for more details.

Property (2.16) implies that the laws Q^n of V^n are uniformly tight on the path space.

Let us now prove that each limiting point Q of this sequence is solution of the nonlinear martingale problem associated with (2.11), i.e., that for (X_t) the canonical process on $\mathbb{D}_T$ and for $\phi \in C_b^1(\mathbb{R}^3)$, $t > 0$,

$$H_t^\phi = \phi(X_t) - \phi(X_0) - \int_0^t \int_0^M \int_0^\pi \int_0^{2\pi} (\phi(X_u + \hat{a}(X_u, w, \theta, \varphi, x))$$

$$- \phi(X_u)) \, Q_u(dw) \, \beta(\theta) \, d\theta \, d\varphi \, dx \, du$$

is a Q-martingale, knowing that the similar quantity $H_t^{n,\phi}$, with $\hat{a}_n$ instead of $\hat{a}$ and Q^n instead of Q, is a Q^n-martingale. The only new difficulty in dimension 3 consists in proving that the function, for $s \leqslant t$,

$$K(X,Y) = \int_s^t \int_0^M \int_0^\pi \int_0^{2\pi} (\phi(X_u + \hat{a}(X_u, Y_u, \theta, \varphi, x)) - \phi(X_u))\, \beta(\theta)\, d\theta\, d\varphi\, dx\, du$$

is continuous on $\mathbb{D}_T \times \mathbb{D}_T$, although a is not Lipschitz continuous. Using the translation invariance of the Lebesgue measure $d\varphi$ and the periodicity of $\hat{a}$ in the variable φ, we write

$$\begin{aligned} &|K(X,Y) - K(X',Y')| \\ &\quad \leqslant M \int_s^t \int_0^\pi \int_0^{2\pi} (|\phi(X_u) - \phi(X'_u)| + |\phi(X_u + \hat{a}(X_u, Y_u, \theta, \varphi)) \\ &\qquad - \phi(X'_u + \hat{a}(X'_u, Y'_u, \theta, \varphi + \varphi_0(X_u - Y_u, X'_u - Y'_u)))|)\, \beta(\theta)\, d\theta\, d\varphi\, du \end{aligned}$$

and thanks to Lemma 2.2, we see that the RHS term tends to 0 when the uniform distance between (X, Y) and (X', Y') tends to 0.

A standard proof allows us to conclude that Q is solution of the nonlinear martingale problem (BMP) associated with (2.11). Moreover, using a representation theorem, we can exhibit an enlarged probability space, on which the canonical process is solution of (2.11) (a similar argument is more detailed in the end of the proof of Theorem 2.6). The property (2.12) follows easily from (2.16). ∎

Let us now prove Theorem 2.6.

Proof. In order to apply Proposition 2.10, we consider some cutoff of the cross-section in both variables.

We introduce the following **approximating model:**

Let $l, k \in \mathbb{N}^*$ and define

$$\beta_l(\theta) = \beta(\theta)\, \mathbf{1}_{|\theta| \geqslant \frac{1}{l}}; \qquad \psi_k(z) = h(|z|)(|z|^\gamma \wedge k), \qquad \forall z \in \mathbb{R}^3.$$

Each function ψ_k is locally Lipschitz continuous and is bounded by kH, where H is a bound of the function h. Thanks to Proposition 2.10 and for each (k, l), there exists a weak solution to the nonlinear stochastic differential equation (SDE_{kl}):

$$\begin{aligned} V_t^{k,l} = V_0 + \int_0^t \int_0^1 \int_{\mathbb{R}_+} \int_0^\pi \int_0^{2\pi} & a(V_{s-}^{k,l}, W_{s-}^{k,l}(\alpha), \theta, \varphi) \\ & \times \mathbf{1}_{\{x \leqslant \psi_k(V_{s-}^{k,l} - W_{s-}^{k,l}(\alpha))\}} N_{k,l}(ds, d\alpha, dx, d\theta, d\varphi) \end{aligned} \tag{2.17}$$

where $N_{k,l}$ is a point Poisson measure with intensity $ds\, d\alpha\, dx\, \beta_l(\theta)\, d\theta\, d\varphi$ on $[0,T]\times[0,1]\times[0,kH]\times[0,\pi]\times[0,2\pi]$. So the associated nonlinear martingale problem $(BMP_{k,l})$ has a solution $P^{k,l}$. The aim is now to prove that the sequence $(P^{k,l})$ of probability measures on the path space $\mathbb{D}_T$ is uniformly tight and that each limit point is solution of the initial nonlinear martingale problem (BMP).

Since the limit case has sense only in a compensated form, we write each equation (2.17) in its compensated form:

$$
\begin{aligned}
V_t^{k,l} &= V_0 - b_l \int_0^t \int_0^1 \psi_k(V_s^{k,l} - W_s^{k,l}(\alpha))(V_s^{k,l} - W_s^{k,l}(\alpha))\, d\alpha\, ds \\
&\quad + \int_0^t \int_0^1 \int_{\mathbb{R}_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}^{k,l}, W_{s-}^{k,l}(\alpha), \theta, \varphi)\, \mathbf{1}_{\{x \leqslant \psi_k(V_{s-}^{k,l} - W_{s-}^{k,l}(\alpha))\}} \\
&\quad \times \tilde{N}_{k,l}(ds, d\alpha, dx, d\theta, d\varphi)
\end{aligned}
$$

where

$$
b_l = \pi \int_0^\pi (1-\cos\theta)\, \beta_l(\theta)\, d\theta.
$$

Lemma 2.11

$$
\sup_{k,l} E(\sup_{t\leqslant T} |V_t^{k,l}|^4) < +\infty. \tag{2.18}
$$

Proof of Lemma 2.11. Thanks to (2.4), we obtain easily that

$$
\begin{aligned}
E(\sup_{s\leqslant t} |V_s^{k,l}|^4) &\leqslant K\left(E(|V_0|^4) + \int_0^t \int_0^1 E(\sup_{u\leqslant s} |V_u^{k,l} - W_u^{k,l}(\alpha)|^4 + 1)\, d\alpha\, ds \right) \\
&\leqslant K\left(1 + \int_0^t E(\sup_{u\leqslant s} |V_u^{k,l}|^4)\, ds \right)
\end{aligned} \tag{2.19}
$$

and the constant number K does not depend on k and l. By Proposition 2.10, $E(\sup_{s\leqslant T} |V_s^{k,l}|^4)$ is finite for each k, l and the proof is obtained by Gronwall's lemma. ∎

It is thus classical to show that the Aldous criterion is satisfied. Hence the sequence $(P^{k,l})$ is tight.

Let us now identify each limit point of $(P^{k,l})$. Let Q be a limit value of this sequence. We consider the compensated martingale problems. Let $(X_t)_t$ be the canonical process on $\mathbb{D}_T$ and for $\phi \in C_b^2(\mathbb{R}^3)$, $t > 0$, we set

$$H_t^\phi = \phi(X_t) - \phi(X_0) + b \int_0^t \int_{w \in \mathbb{R}^2} \nabla\phi(X_u) \cdot (X_u - w)\, \psi(X_u - w)\, Q_u(dw)\, du$$

$$- \int_0^t \int_0^\pi \int_0^{2\pi} \int_{\mathbb{R}^3} (\phi(X_u + a(X_u, w, \theta, \varphi, x)) - \phi(X_u)$$

$$- a(X_u, w, \theta, \varphi, x)(X_u - w) \cdot \nabla\phi(X_u))\, \psi(X_u - w)\, Q_u(dw)\, \beta(\theta)\, d\theta\, d\varphi\, du$$

and $H_t^{k,l,\phi}$ denotes a similar quantity with ψ_k instead of ψ, β_l instead of β, b_l instead of b, and $P_u^{k,l}$ instead of Q_u. The probability measure Q will be a solution of the nonlinear martingale problem (BMP) with initial law Q_0 if it satisfies for each $0 \leqslant s_1 < \cdots < s_p < s < t \leqslant T$, each $G \in C_b((\mathbb{R}^3)^p)$,

$$\langle (H_t^\phi - H_s^\phi)\, G(X_{s_1}, ..., X_{s_p}), Q \rangle = 0. \tag{2.20}$$

Since $P^{k,l}$ is a solution of $(BMP_{k,l})$, we already know that

$$\langle (H_t^{k,l,\phi} - H_s^{k,l,\phi})\, G(X_{s_1}, ..., X_{s_p}), P^{k,l} \rangle = 0.$$

Since the sequence $(P^{k,l})$ satisfies the Aldous criterion, the law Q is the law of a quasi-càg process (cf. ref. 19, p. 321). Then the mapping $F: x \mapsto (\phi(x_t) - \phi(x_s))\, G(x_{s_1}, ..., x_{s_p})$ is Q-a.e. continuous and bounded from $\mathbb{D}_T$ to $\mathbb{R}$. Thus $\langle F, P^{k,l} \rangle$ tends to $\langle F, Q \rangle$ as k, l tend to infinity.

Now, let us successively prove that

$$T_1 = \left\langle \left(\int_s^t \int_0^\pi \int_0^{2\pi} \int_{\mathbb{R}^3} (\phi(X_u + a(X_u, w, \theta, \varphi)) - \phi(X_u) - a(X_u, w, \theta, \varphi) \cdot \nabla\phi(X_u)) \right.\right.$$

$$\left.\left. \times (\psi(X_u - w) - \psi_k(X_u - w))\, P_u^{k,l}(dw)\, \beta_l(\theta)\, d\theta\, d\varphi\, du \right) G(X_{s_1}, ..., X_{s_p}), P^{k,l} \right\rangle,$$

$$T_2 = \left\langle \left(\int_s^t \int_0^\pi \int_0^{2\pi} \int_{\mathbb{R}^3} (\phi(X_u + a(X_u, w, \theta, \varphi)) - \phi(X_u) - a(X_u, w, \theta, \varphi) \cdot \nabla\phi(X_u)) \right.\right.$$

$$\left.\left. \times \psi(X_u - w)(\beta_l(\theta) - \beta(\theta))\, P_u^{k,l}(dw)\, d\theta\, d\varphi\, du \right) G(X_{s_1}, ..., X_{s_p}), P^{k,l} \right\rangle,$$

$$T_3 = \left\langle G(X_{s_1}, ..., X_{s_p}) \int_s^t \int_{\mathbb{R}^3} K_{\beta,\gamma}^\phi(X_u, Y_u), P^{k,l}(dX) \otimes P^{k,l}(dY) \right\rangle$$

$$- \left\langle G(X_{s_1}, ..., X_{s_p}) \int_s^t \int_{\mathbb{R}^3} K_{\beta,\gamma}^\phi(X_u, Y_u), Q(dX) \otimes Q(dY) \right\rangle,$$

and the term T_4 similar to T_1 corresponding to the drift term, tend to 0 as k, l tend to infinity.

Term T_1:

$$|T_1| \leqslant K \int_0^\pi \theta^2 \beta_l(\theta)\, d\theta \left\langle \int_s^t \int_{\mathbb{R}^3} |X_u - w|^2 \left(|X_u - w|^\gamma - (|X_u - w|^\gamma) \wedge k\right) \right.$$

$$\left. \times P_u^{k,l}(dw)\, du,\, P^{k,l} \right\rangle$$

$$\leqslant K \left\langle \int_s^t \int_{\mathbb{R}^3} |X_u - w|^{2+\gamma}\, \mathbf{1}_{\{|X_u - w|^\gamma \geqslant k\}} P_u^{k,l}(dw)\, du,\, P^{k,l} \right\rangle$$

$$\leqslant K \left\langle \int_s^t \int_{\mathbb{R}^3} |X_u - w|^{2+\gamma}\, \mathbf{1}_{\{|X_u - w| \leqslant (k)^{\frac{1}{\gamma}}\}} P_u^{k,l}(dw)\, du,\, P^{k,l} \right\rangle$$

$$\leqslant K(k)^{\frac{2+\gamma}{\gamma}}$$

and T_1 tends to zero when k tends to infinity, uniformly in l since $\int_0^\pi \theta^2 \beta_l(\theta)\, d\theta \leqslant \int_0^\pi \theta^2 \beta(\theta)\, d\theta < +\infty$, and since $\frac{2+\gamma}{\gamma} < 0$.

Term T_4: By a similar study with the drift term, we obtain

$$|T_4| \leqslant K \left\langle \int_s^t \int_{\mathbb{R}^3} |X_u - w| \left(|X_u - w|^\gamma - (|X_u - w|^\gamma) \wedge k\right) P_u^{k,l}(dw)\, du,\, P^{k,l} \right\rangle$$

$$\leqslant K(k)^{\frac{1+\gamma}{\gamma}}$$

and T_4 tends to zero when k tends to infinity, since $\gamma \in (-1, 0]$.

Term T_2:

$$|T_2| \leqslant K \int_0^\pi \theta^2\, |\beta_l(\theta) - \beta(\theta)|\, d\theta \left\langle \int_s^t \int_{\mathbb{R}^3} (|X_u - w|^{2+\gamma})\, P_u^{k,l}(dw)\, du,\, P^{k,l} \right\rangle$$

$$\leqslant K(\sup_{k,l} E_{P^{k,l}}(\sup_{u \leqslant T} |X_u|^2) + 1) \int_{-\pi}^\pi \theta^2\, |\beta_l(\theta) - \beta(\theta)|\, d\theta$$

which tends to 0 as l tends to infinity, uniformly in k thanks to Lemma 2.11.

Term T_3: Let us define the function $F(x, y)$ on $\mathbb{D}_T \times \mathbb{D}_T$ by $F(x, y) = \int_s^t K_{\beta,\gamma}^\phi(x_u, y_u)\, du$. The function F is $Q \otimes Q$-a.e. continuous by a similar argument as in the proof of Proposition 2.10 and is not bounded. Estimates (2.4) imply that

$$|F(x, y)| \leqslant K \int_0^\pi \theta^2 \beta(\theta)\, d\theta \big(\sup_{s \leqslant u \leqslant t} (|x_u - y_u|^{2+\gamma} + |x_u - y_u|^{1+\gamma})\big)$$
$$\leqslant K\big(\sup_{u \leqslant T} |x_u|^2 + \sup_{u \leqslant T} |y_u|^2 + 1\big).$$

Now, the measure $P^{k,l} \otimes P^{k,l}$ converges obviously to $Q \otimes Q$. Then, for each fixed real positive number C, the sequence $\langle F \wedge C, P^{k,l} \otimes P^{k,l} \rangle$ converges to $\langle F \wedge C, Q \otimes Q \rangle$. We remark that

$$|F(x, y)|\, \mathbf{1}_{\{|F(x, y)| \geqslant C\}}$$
$$\leqslant K\big(\sup_{u \leqslant T} |x(u)|^2 + \sup_{u \leqslant T} |y(u)|^2 + 1\big)\, \mathbf{1}_{\{\sup_{u \leqslant T} |x(u)|^2 + \sup_{u \leqslant T} |y(u)|^2 \geqslant C/K - 1\}}$$
$$\leqslant K\big(\sup_{u \leqslant T} |x(u)|^2 + \sup_{u \leqslant T} |y(u)|^2 + 1\big)$$
$$\times \big(\mathbf{1}_{\{\sup_{u \leqslant T} |x(u)|^2 \geqslant C/2K - 1/2\}} + \mathbf{1}_{\{\sup_{u \leqslant T} |y(u)|^2 \geqslant C/2K - 1/2\}}\big)$$

and it is easy to prove thanks to Lemma 2.11 that

$$\sup_{k,l} \big\langle \big(\sup_{u \leqslant T} |x(u)|^2 + \sup_{u \leqslant T} |y(u)|^2 + 1\big)$$
$$\times \big(\mathbf{1}_{\{\sup_{u \leqslant T} |x(u)|^2 \geqslant C/2K - 1/2\}} + \mathbf{1}_{\{\sup_{u \leqslant T} |y(u)|^2 \geqslant C/2K - 1/2\}}\big), P^{k,l} \otimes P^{k,l} \big\rangle$$

tends to 0 as C tends to infinity.

We have thus proved that each limit law of the sequence $(P^{k,l})$ is solution of the martingale problem (BMP). Since such limits exist thanks to the Aldous criterion, we deduce obviously from this approach the existence of at least one solution to (BMP).

Let us now show that each solution Q of (BMP) is a weak solution of (SDE).

The canonical process X is a semimartingale under Q. Then a comparison between the Itô formula and the martingale problem proves that X is a pure jump process and that its Lévy measure is the image measure of the measure m on $[0, T] \times [0, 1] \times \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi]$ by the mapping $(s, \alpha, x, \theta, \varphi) \mapsto a(X_{s-}, W_{s-}(\alpha), \theta, \varphi)\, \mathbf{1}_{\{x \leqslant \psi(X_{s-} - W_{s-}(\alpha))\}}$. Then by a representation theorem for point measures,[8] there exist on an enlarged probability space a square integrable variable V_0 and a Poisson-point measure N with intensity m such that (X, W, N, V_0) is a solution of (SDE). ∎

3. CONVERGENCE OF RENORMALIZED BOLTZMANN PROCESSES TOWARDS A LANDAU PROCESS

3.1. A Probabilistic Interpretation of the Landau Equation

The Landau equation, also called the Fokker–Planck–Landau equation, describes the collisions of particles in a plasma and is obtained as limit of Boltzmann equations when the collisions become grazing. In the spatially homogeneous case, it writes in $\mathbb{R}^3$:

$$\frac{\partial f}{\partial t} = Q_L(f, f) \tag{3.1}$$

with

$$Q_L(f, f)(t, v) = \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial}{\partial v_i} \left\{ \int_{\mathbb{R}^3} dv_*\, A_{ij}(v - v_*) \left[f(t, v_*)\, \frac{\partial f}{\partial v_j}(t, v) - f(t, v)\, \frac{\partial f}{\partial v_{*j}}(t, v_*) \right] \right\}$$

where $f(t, v) \geqslant 0$ is the density of particles having velocity $v \in \mathbb{R}^3$ at time $t \in \mathbb{R}^+$, and $(A_{ij}(z))_{1 \leqslant i, j \leqslant 3}$ is a nonnegative symmetric matrix depending on the interaction between the particles, of the form

$$\begin{aligned} A(z) &= \Lambda\, |z|^{\gamma+2}\, \Pi(z)\, h(|z|) \\ &= \Lambda\, |z|^{\gamma}\, h(|z|) \begin{bmatrix} z_2^2 + z_3^2 & -z_1 z_2 & -z_1 z_3 \\ -z_1 z_2 & z_1^2 + z_3^2 & -z_2 z_3 \\ -z_1 z_3 & -z_2 z_3 & z_1^2 + z_2^2 \end{bmatrix} \end{aligned} \tag{3.2}$$

where $\Pi(z)$ is the orthogonal projection on $(z)^{\perp}$, Λ is a positive constant and h is a nonnegative locally Lipschitz continuous bounded function.

By integration by parts, see ref. 27, a weak formulation of the equation (3.1) writes, at least formally, for any test function $\phi \in C_b^2(\mathbb{R}^3)$,

$$\begin{aligned} \frac{d}{dt} \int \phi(v)\, f(t, v)\, dv &= \frac{1}{4} \sum_{i,j=1}^{3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dv\, dv_*\, f(t, v)\, f(t, v_*)\, A_{ij}(v - v_*)(\partial_{ij}^2 \phi(v) + \partial_{ij}^2 \phi(v_*)) \\ &\quad + \frac{1}{2} \sum_{i=1}^{3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dv\, dv_*\, f(t, v)\, f(t, v_*)\, b_i(v - v_*)(\partial_i \phi(v) - \partial_i \phi(v_*)) \end{aligned} \tag{3.3}$$

where $b_i(z) = \sum_{j=1}^{3} \partial_j A_{ij}(z) = -2\Lambda h(|z|)\, |z|^{\gamma}\, z_i$.

As for the Boltzmann equation, the equation (3.3) conserves the mass, thus we give a definition of probability-measure solutions of the Landau equation:

Definition 3.1. Let P_0 belong to $\mathscr{P}_2(\mathbb{R}^3)$. A probability measure solution of the Landau equation (3.4) with initial data P_0 is a probability measure family $(P_t)_{t \geqslant 0}$ on $\mathbb{R}^3$ satisfying

$$\langle \phi, P_t \rangle = \langle \phi, P_0 \rangle + \int_o^t \langle L^{\phi}(v, v_*), P_s(dv)\, P_s(dv_*) \rangle \, ds \tag{3.4}$$

for any function $\phi \in C_b^2(\mathbb{R}^3)$ where L^{ϕ} is the Landau kernel defined on $\mathbb{R}^3 \times \mathbb{R}^3$ by:

$$\begin{aligned} L^{\phi}(v, v_*) &= \frac{1}{2} \sum_{i, j=1}^{3} \partial_{ij}^2 \phi(v)\, A_{ij}(v - v_*) + \sum_{i=1}^{3} \partial_i \phi(v)\, b_i(v - v_*) \\ &= \frac{1}{2} J_{\phi}(v) \colon A(v - v_*) + b(v - v_*) \cdot \nabla \phi(v) \end{aligned}$$

with $J_{\phi} = (\partial_{ij}^2 \phi)_{1 \leqslant i, j \leqslant 3}$.

We now consider the martingale problem associated with the Landau equation and defined as follows.

Definition 3.2. Let P_0 belong to $\mathscr{P}_2(\mathbb{R}^3)$.

Let $(Y_s)_{s \geqslant 0}$ be the canonical process on $\mathscr{C}_T$. A probability measure $P \in \mathscr{P}(\mathscr{C}_T)$ is a solution of the martingale problem (LMP) with initial data P_0 if the law of Y_0 is P_0 and if for any $\phi \in C^2(\mathbb{R}^3)$,

$$\phi(Y_t) - \phi(Y_0) - \int_0^t \int_{\mathbb{R}^3} L^{\phi}(Y_s, v_*)\, P_s(dv_*)\, ds$$

is a P-martingale, where $P_s = P \circ Y_s^{-1}$.

Remark 3.3. The time-marginal family of a solution of the martingale problem (LMP) is a measure-solution of the Fokker–Planck–Landau equation.

Guérin already built in ref. 16, by a direct probabilistic approach, a Landau process solution of a nonlinear stochastic differential equation driven by a white noise and deduced the existence of a measure-solution of the Landau equation for any dimension $\geqslant 2$ and for $\gamma \in (-1, 0]$. We obtain here a new proof of the existence of a solution to the Landau process (and then of a solution to the Landau equation) as limit of Boltzmann processes.

3.2. Asymptotic of Boltzmann Processes Towards a Landau Process

We are now interested in stating the convergence in law of Boltzmann processes with "moderately soft potentials," obtained in Section 2, towards a Landau process, when the collisions become grazing. With this aim in view, we consider cross-sections β^ε depending on the grazing collision parameter ε, as in Villani.[27] The function β^ε from $[0, \pi]$ to $\mathbb{R}^+$ satisfies

$$\forall \theta_0 > 0 \;\; \beta^\varepsilon(\theta) \xrightarrow[\varepsilon \to 0]{} 0 \quad \text{uniformly on } \theta \geqslant \theta_0 \tag{3.5}$$

$$\Lambda^\varepsilon = \pi \int_0^\pi \sin^2(\theta/2)\, \beta^\varepsilon(\theta)\, d\theta \xrightarrow[\varepsilon \to 0]{} \Lambda > 0 \tag{3.6}$$

Let us remark that these hypotheses contain the case $\beta^\varepsilon(\theta) = \frac{1}{|\log \varepsilon|} \times \frac{\cos(\theta/2)}{\sin^3(\theta/2)} \mathbb{I}_{\theta \geqslant \varepsilon}$ introduced by Degond and Lucquin-Desreux[4] for a Coulomb potential ($\gamma = -3$) and $\beta^\varepsilon(\theta) = \frac{1}{\varepsilon^3} \beta(\frac{\theta}{\varepsilon})$ introduced by Desvillettes[5] for non Coulomb potentials.

Let us notice that

Lemma 3.4

(1) $\int_0^\pi \beta^\varepsilon(\theta)\, d\theta \xrightarrow[\varepsilon \to 0]{} +\infty$,

(2) For $k \geqslant 3$,

$$\int_0^\pi \sin^k(\theta/2)\, \beta^\varepsilon(\theta)\, d\theta \xrightarrow[\varepsilon \to 0]{} 0.$$

The proof is left to the reader.

For each $\varepsilon > 0$, for $\gamma \in (-1, 0]$, we define the Boltzmann kernel $K^\phi_{\beta^\varepsilon, \gamma}$ on $\mathbb{R}^3 \times \mathbb{R}^3$, as in (2.8), by

$$\begin{aligned} K^\phi_{\beta^\varepsilon, \gamma}(v, v_*) = {} & -b^\varepsilon \psi(v - v_*)(v - v_*) \cdot \nabla\phi(v) + \int_0^{2\pi} \int_0^\pi (\phi(v + a(v, v_*, \theta, \varphi)) \\ & - \phi(v) - a(v, v_*, \theta, \varphi) \cdot \nabla\phi(v))\, \psi(v - v_*)\, \beta^\varepsilon(\theta)\, d\theta\, d\varphi \end{aligned} \tag{3.7}$$

with $b^\varepsilon = \pi \int_0^\pi (1 - \cos\theta)\, \beta^\varepsilon(\theta)\, d\theta$.

We notice that the Boltzmann kernels converge towards the Landau kernel when $\varepsilon \to 0$, for any $v, v_* \in \mathbb{R}^3$ and $\phi \in C_b^2(\mathbb{R}^3)$ (for more details, see the convergence of the term E_1 in Section 3.4).

We denote by $(B^\varepsilon MP)$ the martingale problem associated with the Boltzmann equation defined as in Definition 2.4 replacing $K^\phi_{\beta,\gamma}$ with $K^\phi_{\beta^\varepsilon,\gamma}$. In the previous section, we have proved the existence of a solution Q^ε of $(B^\varepsilon MP)$. We are now interested in the asymptotic behaviour of the sequence $(Q^\varepsilon)_{\varepsilon>0}$ when ε tends to 0.

We state the following main theorem.

Theorem 3.5. Consider a bounded locally Lipschitz continuous nonnegative function h, $\gamma \in (-1, 0]$, β^ε satisfying (3.5) and (3.6) and Q_0 a finite fourth-order moment probability measure. Let $Q^\varepsilon \in \mathscr{P}(\mathbb{D}_T)$ be a solution of the nonlinear martingale problem $(B^\varepsilon MP)$ with kernel $K_{\beta^\varepsilon,\gamma}$ defined by (3.7) and initial data Q_0.

Then the sequence $(Q^\varepsilon)_{\varepsilon>0}$ is tight when ε tends to 0, and any of its subsequences converges towards a solution $P \in \mathscr{P}(\mathscr{C}_T)$ of the nonlinear martingale problem (LMP), associated with the Landau equation (3.4) having diffusion matrix defined by (3.2), with initial law Q_0.

Remark 3.6. When $\gamma = 0$ and under some regularity assumptions on h, Guérin has proved in ref. 17, Corollary 7 the uniqueness of a solution P to the martingale problem (LMP). Then, in this case, the sequence $(Q^\varepsilon)_{\varepsilon>0}$ converges towards this unique solution P.

Let us notice that Villani[27] and Goudon[13] proved the existence of weak function solutions of the Landau equation for soft potentials using the convergence of the solutions of the Boltzmann equation towards the solutions of the Landau equation. The interest of our approach is the understanding of this convergence at the microscopic level of processes. When ε decreases, the Boltzmann processes jump more and more often with smaller jumps, and then finally converge to a (continuous) diffusion process. Moreover, our convergence result is true for general (even degenerate as Dirac measures) initial data and leads naturally to particle approximations.

3.3. *C*-Tightness of the Sequence $(Q^\varepsilon)_{\varepsilon>0}$

We assume that Q_0 has a finite fourth-order moment.

Let Q^ε be a solution of the martingale problem $(B^\varepsilon MP)$ obtained in Theorem 2.6 and X the canonical process on $\mathbb{D}_T$. Thanks to the point 2) of Theorem 2.6, for any $\varepsilon > 0$, the probability Q^ε satisfies $E_{Q^\varepsilon}(\sup_{0\leqslant t\leqslant T} |X_t|^4) \leqslant K^\varepsilon$ with K^ε a positive constant depending on ε only through $\int_{-\pi}^{\pi} \sin^4(\theta/2) \times \beta^\varepsilon(\theta)\, d\theta$, $\int_{-\pi}^{\pi} \sin^2(\theta/2)\, \beta^\varepsilon(\theta)\, d\theta$ and b^ε according to Lemma 2.2. Using

Lemma 3.4 and (3.6), we notice that the sequence $(K^{\varepsilon})_{\varepsilon>0}$ converges as ε tends to 0. Then there exists $K>0$ such that

$$\sup_{\varepsilon>0} E_{Q^{\varepsilon}}(\sup_{0\leqslant t\leqslant T} |X_t|^4)\leqslant K \tag{3.8}$$

Thanks to the Aldous criterion, we deduce, with similar arguments as in Section 2, that the sequence $(Q^{\varepsilon})_{\varepsilon>0}$ is tight in $\mathscr{P}(\mathbb{D}_T)$, and then each limiting point P of $(Q^{\varepsilon})_{\varepsilon>0}$ belongs to $\mathscr{P}(\mathbb{D}_T)$.

We now prove that the sequence $(Q^{\varepsilon})_{\varepsilon>0}$ is moreover C-tight, in the sense of Jacod and Shiryaev,[19] p. 315, and then P will belong to $\mathscr{P}(\mathscr{C}_T)$.

As the sequence $(Q^{\varepsilon})_{\varepsilon>0}$ is tight and according to ref. 19, Proposition 3.26 (iii), we just have to prove that for any $\eta>0$, for $\Delta X_t=X_t-X_{t^-}$,

$$\lim_{\varepsilon\to 0} Q^{\varepsilon}(\sup_{t\leqslant T} |\Delta X_t|>\eta)=0.$$

We use the stochastic differential equation (SDE) introduced in Section 2.2. Let V^{ε} be a process with distribution Q^{ε} such that

$$\begin{aligned}V_t^{\varepsilon}&=V_0-b^{\varepsilon}\int_0^t\int_0^1 \psi(V_s^{\varepsilon}-W_s^{\varepsilon}(\alpha))(V_s^{\varepsilon}-W_s^{\varepsilon}(\alpha))\,d\alpha\,ds\\&\quad+\int_0^t\int_0^1\int_{\mathbb{R}_+}\int_0^{\pi}\int_0^{2\pi} a(V_{s-}^{\varepsilon},W_{s-}^{\varepsilon}(\alpha),\theta,\varphi)\,\mathbf{1}_{\{x\leqslant\psi(V_{s-}^{\varepsilon}-W_{s-}^{\varepsilon}(\alpha))\}}\\&\quad\times\tilde{N}^{\varepsilon}(ds,d\alpha,dx,d\theta,d\varphi)\end{aligned}$$

with $\mathscr{L}_{\alpha}(W^{\varepsilon})=\mathscr{L}(V^{\varepsilon})=Q^{\varepsilon}$ and $\tilde{N}^{\varepsilon}(ds,d\alpha,dx,d\theta,d\varphi)$ is the compensated martingale of a Poisson measure with intensity $m^{\varepsilon}(dt,d\alpha,dx,d\theta,d\varphi)=dt\,d\alpha\,dx\,\beta^{\varepsilon}(\theta)\,d\theta\,d\varphi$.

Then, by Tchebychev and Burkholder–Davis–Gundy inequalities for jump semimartingales and Lemma 2.2,

$$\begin{aligned}&Q^{\varepsilon}(\sup_{t\leqslant T}|\Delta X_t|>\eta)\\&\quad\leqslant\frac{1}{\eta^4}E(\sup_{t\leqslant T}|\Delta V_t^{\varepsilon}|^4)\leqslant\frac{1}{\eta^4}E\left(\sum_{t\leqslant T}|\Delta V_t^{\varepsilon}|^4\right)\\&\quad\leqslant\frac{1}{\eta^4}E\left(\int_0^T\int_0^1\int_0^{\pi}\int_0^{2\pi}|a(V_{s-}^{\varepsilon},W_{s-}^{\varepsilon}(\alpha),\theta,\varphi)|^4\,\psi(V_{s-}^{\varepsilon}-W_{s-}^{\varepsilon}(\alpha))\right.\\&\qquad\left.\times\beta^{\varepsilon}(\theta)\,d\theta\,d\varphi\,d\alpha\,ds\right)\\&\quad\leqslant\frac{K}{\eta^4}\left(\int_0^T\int_0^1 E(|V_{u-}^{\varepsilon}-W_{u-}^{\varepsilon}(\alpha)|^{\gamma+4})\,d\alpha\,du\right)\int_0^{\pi}|\sin(\theta/2)|^4\,\beta^{\varepsilon}(\theta)\,d\theta\end{aligned}$$

with K independent of ε. Thanks to estimates (2.4) and (3.8), we obtain

$$Q^{\varepsilon}(\sup_{t\leqslant T} |\Delta X_t| > \eta) \leqslant \frac{KT}{\eta^4} \int_0^{\pi} |\sin(\theta/2)|^4 \beta^{\varepsilon}(\theta)\, d\theta$$

As $\int_0^{\pi} |\sin(\theta/2)|^4 \beta^{\varepsilon}(\theta)\, d\theta$ tends to 0 as ε tends to 0, the sequence $(Q^{\varepsilon})_{\varepsilon>0}$ is C-tight.

3.4. Identification of the Limit Point Values *P*

Let P be a limiting value of the sequence (Q^{ε}). Then P is the limit of a subsequence (Q^{ε}) that we will still denote by (Q^{ε}) for simplicity. We wish to prove that P is a solution of the martingale problem (LMP). Let $\phi \in C_b^2(\mathbb{R}^3)$. We define the two following processes on $\mathbb{D}_T$

$$M_t^{\varepsilon} = \phi(X_t) - \phi(X_0) - \int_0^t \int_{\mathbb{R}^3} K^{\phi}_{\beta^{\varepsilon},\gamma}(X_s, v_*)\, Q_s^{\varepsilon}(dv_*)\, ds \tag{3.9}$$

$$M_t = \phi(X_t) - \phi(X_0) - \int_0^t \int_{\mathbb{R}^3} L^{\phi}(X_s, v_*)\, P_s(dv_*)\, ds \tag{3.10}$$

The probability measure P will be a solution of the nonlinear martingale problem (LMP) with initial law Q_0 if it satisfies, for any $0 \leqslant s_1 < \cdots < s_p < s < t \leqslant T$ and $G \in C_b((\mathbb{R}^3)^p)$,

$$\langle (M_t - M_s)\, G(X_{s_1},..., X_{s_p}), P\rangle = 0$$

However, Q^{ε} is a solution of $(B^{\varepsilon}MP)$, then, for any $0 \leqslant s_1 < \cdots < s_p < s < t \leqslant T$ and $G \in C_b((\mathbb{R}^3)^p)$,

$$\langle (M_t^{\varepsilon} - M_s^{\varepsilon})\, G(X_{s_1},..., X_{s_p}), Q^{\varepsilon}\rangle = 0$$

Thus, we want to state the following convergence

$$E_{Q^{\varepsilon}}((M_t^{\varepsilon} - M_s^{\varepsilon})\, G(X_{s_1},..., X_{s_p})) \xrightarrow[\varepsilon\to 0]{?} E_P((M_t - M_s)\, G(X_{s_1},..., X_{s_p}))$$

1. Since (Q^{ε}) is C-tight, the distribution P charges only the set $\mathscr{C}_T$, then the mapping $F: x \mapsto (\phi(x_t) - \phi(x_s))\, G(x_{s_1},..., x_{s_p})$ is P-continuous and bounded from $\mathbb{D}_T$ to $\mathbb{R}$. Thus $\langle F, Q^{\varepsilon}\rangle$ tends to $\langle F, P\rangle$ as ε tends to zero.

2. We now study the convergence of the term

$$E_{Q^\varepsilon}\left(\left\{\int_s^t \int_{\mathbb{R}^3} K^\phi_{\beta^\varepsilon,\gamma}(X_u, v_*)\, Q^\varepsilon_u(dv_*)\, du\right\} G(X_{s_1},..., X_{s_p})\right) \quad \text{to}$$

$$E_P\left(\left\{\int_s^t \int_{\mathbb{R}^3} L^\phi(X_u, v_*)\, P_u(dv_*)\, du\right\} G(X_{s_1},..., X_{s_p})\right).$$

If we denote by (X, Y) the canonical process on $\mathbb{D}_T \times \mathbb{D}_T$, we can write

$$\begin{aligned} E_1 + E_2 = E_{Q^\varepsilon}&\left(\left\{\int_s^t \langle K^\phi_{\beta^\varepsilon,\gamma}(X_u, v_*), Q^\varepsilon_u(dv_*)\rangle\, du\right\} G(X_{s_1},..., X_{s_p})\right) \\ -E_P&\left(\left\{\int_s^t \langle L^\phi(X_u, v_*), P_u(dv_*)\rangle\, du\right\} G(X_{s_1},..., X_{s_p})\right) \end{aligned}$$

with

$$E_1 = E_{Q^\varepsilon\otimes Q^\varepsilon}\left(\left\{\int_s^t (K^\phi_{\beta^\varepsilon,\gamma}(X_u, Y_u) - L^\phi(X_u, Y_u))\, du\right\} G(X_{s_1},..., X_{s_p})\right)$$

$$\begin{aligned} E_2 = E_{Q^\varepsilon\otimes Q^\varepsilon}&\left(\left\{\int_s^t L^\phi(X_u, Y_u)\, du\right\} G(X_{s_1},..., X_{s_p})\right) \\ -E_{P\otimes P}&\left(\left\{\int_s^t L^\phi(X_u, Y_u)\, du\right\} G(X_{s_1},..., X_{s_p})\right) \end{aligned}$$

(a) Study of E_1

$$|E_1| \leqslant K E_{Q^\varepsilon\otimes Q^\varepsilon}\left(\int_s^t |K^\phi_{\beta^\varepsilon,\gamma}(X_u, Y_u) - L^\phi(X_u, Y_u)|\, du\right) \tag{3.11}$$

The Taylor development of ϕ writes

$$\phi(v+u) = \phi(v) + u\cdot\nabla\phi(v) + \tfrac{1}{2}\, u^t\cdot J_\phi(v)\cdot u + O(|u|^3)$$

We notice that $u^t\cdot J_\phi(v)\cdot u = J_\phi(v) : u\cdot u^t$. Then we divide the expectation of the right term in (3.11) in three parts:

$$E_{Q^\varepsilon\otimes Q^\varepsilon}\left(\int_s^t |K^\phi_{\beta^\varepsilon,\gamma}(X_u, Y_u) - L^\phi(X_u, Y_u)|\, du\right) \leqslant E_{11} + E_{12} + E_{13}$$

with

$$E_{11}=K(|-2\Lambda+b^{\varepsilon}|)\,E_{Q^{\varepsilon}\otimes Q^{\varepsilon}}\left(\int_s^t |\psi(X_u-Y_u)(X_u-Y_u)\cdot\nabla\phi(X_u)|\,du\right)$$

$$E_{12}=KE_{Q^{\varepsilon}\otimes Q^{\varepsilon}}\left(\int_s^t (\psi(X_u-Y_u))\left|\left(J_\phi(X_u)\colon \Lambda|X_u-Y_u|^2\,\Pi(X_u-Y_u)\right.\right.\right.$$
$$\left.\left.\left.-\int_0^{2\pi}\int_0^{\pi} a(X_u,Y_u,\theta,\varphi)\cdot a^t(X_u,Y_u,\theta,\varphi)\,\beta^{\varepsilon}(\theta)\,d\theta\,d\varphi\right)\right|\,du\right)$$

$$E_{13}=KE_{Q^{\varepsilon}\otimes Q^{\varepsilon}}\left(\int_s^t \psi(X_u-Y_u)\left(\int_0^{2\pi}\int_0^{\pi}|a(X_u,Y_u,\theta,\varphi)|^3\,\beta^{\varepsilon}(\theta)\,d\theta\,d\varphi\right)du\right)$$

- Using estimates (2.4) and thanks to (3.8), we get

$$E_{11}\leqslant K\,|-2\Lambda+b^{\varepsilon}|$$

As $b^{\varepsilon}\xrightarrow[\varepsilon\to 0]{} 2\Lambda$, E_{11} converges towards 0 as ε tends to 0.

- Let us now study E_{12}. After some computations, we prove that

$$\int_0^{2\pi} a(X_u,Y_u,\theta,\varphi)\cdot a^t(X_u,Y_u,\theta,\varphi)\,d\varphi$$
$$=\frac{\pi}{4}\,[\Pi(X_u-Y_u)\sin^2\theta+2(I-\Pi(X_u-Y_u))(\cos\theta-1)^2]\,|X_u-Y_u|^2$$

Then

$$\int_0^{2\pi}\int_0^{\pi} a(X_u,Y_u,\theta,\varphi)\cdot a^t(X_u,Y_u,\theta,\varphi)\,\beta^{\varepsilon}(\theta)\,d\theta\,d\varphi$$
$$\xrightarrow[\varepsilon\to 0]{}\Lambda\Pi(X_u-Y_u)\,|X_u-Y_u|^2$$

Thanks to (3.8), we conclude that E_{12} converges towards 0 as ε tends to 0.

- Using similar arguments and Lemma 2.2, we prove the same convergence for E_{13}.

Finally, we have proved that $E_1\xrightarrow[\varepsilon\to 0]{} 0$.

(b) Study of E_2

The functions $f_{ij}^{A}\colon \mathbb{D}_T\times\mathbb{D}_T\to\mathbb{R}$, $(x,y)\mapsto G(X_{s_1},\ldots,X_{s_p})\int_s^t A_{ij}(x_u-y_u)\times\partial_{ij}\phi(x_u)\,du$ and $f_i^{b}\colon \mathbb{D}_T\times\mathbb{D}_T\to\mathbb{R}$, $(x,y)\mapsto G(X_{s_1},\ldots,X_{s_p})\int_s^t b_i(x_u-y_u)\times\partial_i\phi(x_u)\,du$ are continuous functions ($\gamma\in(-1,0]$), but not necessarily bounded. Nevertheless, using similar arguments as in the proof of Theorem 2.6 in Section 2, we obtain $E_2 \xrightarrow[\varepsilon \to 0]{} 0$.

Conclusion. For any $(t, s, s_1,..., s_p) \in (\mathbb{R}_+)^{p+2}$, with $0 \leqslant s_1 \leqslant \cdots \leqslant s_p \leqslant s < t$, we have proved that

$$E_{Q^\varepsilon}((M_t - M_s)\, G(X_{s_1},..., X_{s_p})) \xrightarrow[n \to \infty]{} E_P((M_t - M_s)\, G(X_{s_1},..., X_{s_p}))$$

which implies that

$$E_P((M_t - M_s)\, G(X_{s_1},..., X_{s_p})) = 0$$

So, $(M_t)_{t \geqslant 0}$ is a P-martingale and P satisfies the martingale problem (LMP).

4. A STOCHASTIC PARTICLE APPROXIMATION

Our aim here is the construction of some simulable stochastic colliding particle systems converging in a certain sense to the law of a Landau process. More precisely we consider cutoff cross-sections (to obtain simulable systems) depending on a grazing collision parameter ε. We define the interacting particle systems by a Monte-Carlo approach, consisting in replacing the nonlinearity with the empirical measure of the system. These particle systems will conserve the momentum and kinetic energy. For a fixed ε, it has already been proved that when the parameter of the cutoff and the size of the system tend to infinity, the empirical measures of the system tend to the law of a Boltzmann process (see, for example, ref. 12). The novelty here is that this convergence is uniform in the parameter ε. If moreover ε tends to 0, we observe the transition from Boltzmann to Landau equations on the particle system. This result will be exploited in the last section to construct an efficient Monte-Carlo algorithm allowing to see this transition.

We consider the sequence of cutoff cross-sections

$$B_{k,\varepsilon}(z, \theta) = \psi_k(z)\, \beta^\varepsilon(\theta) \tag{4.1}$$

where $\psi_k(z) = h(|z|)(|z|^\gamma \wedge k)$, h is a locally Lipschitz function bounded by H, $\gamma \in (-1, 0]$, ε a parameter tending to 0, β^ε is a $L^1([0, \pi])$-function satisfying (3.5) and (3.6), k is a positive integer.

In order to define the interacting systems, we will "replace" the nonlinearity in (2.11) with the empirical measure of the system. Hence we introduce a family of independent Poisson-point measures $(N^{\varepsilon, ij})_{1 \leqslant i < j \leqslant n}$ on $[0, \pi] \times [0, 2\pi] \times [0, kH] \times [0, T]$ with intensities $\frac{1}{n-1}\, \beta^\varepsilon(\theta)\, d\theta\, d\varphi\, dx\, dt$.

For $i > j$, we set $N^{\varepsilon, ij} = N^{\varepsilon, ji}$ (we thus choose a binary mean-field interaction, close to the physical interpretation). We define the process $(V^{k\varepsilon, in})_{1 \leqslant i \leqslant n}$ solution of the following stochastic differential system:

$$V_t^{k\varepsilon, in} = V_0^i + \sum_{j \neq i, j=1}^{n} \int_0^t \int_0^{kH} \int_0^{2\pi} \int_0^{2\pi} a(V_{s-}^{k\varepsilon, in}, V_{s-}^{k\varepsilon, jn}, \theta, \varphi)$$

$$\times \mathbf{1}_{\{x \leqslant \psi_k(V_{s-}^{k\varepsilon, in} - V_{s-}^{k\varepsilon, jn})\}} N^{\varepsilon, ij}(d\theta, d\varphi, dx, ds). \tag{4.2}$$

We construct it easily, working recursively on each interjump interval of the point process $(N^{\varepsilon, ij})_{1 \leqslant i, j \leqslant n}$. The equations are not compensated since for a fixed ε, the function β^ε belongs to $L^1([0, \pi])$. The system conserves momentum and kinetic energy and is a $(\mathbb{R}^3)^n$-valued pure-jump Markov process with the generator defined for $\phi \in C_b((\mathbb{R}^3)^n)$ by

$$\frac{1}{n-1} \sum_{1 \leqslant i, j \leqslant n} \int_0^{2\pi} \int_0^{\pi} \int_0^{kH} \frac{1}{2} (\phi(v^n + \mathbf{e}_i \cdot a(v_i, v_j, \theta, \varphi)\, \mathbf{1}_{\{x \leqslant \psi_k(v_i - v_j)\}}$$

$$+ \mathbf{e}_j \cdot a(v_j, v_i, \theta, \varphi)\, \mathbf{1}_{\{x \leqslant \psi_k(v_i - v_j)\}}) - \phi(v^n))\, dx\, \beta^\varepsilon(\theta)\, d\theta\, d\varphi. \tag{4.3}$$

Here $v^n = (v_1,..., v_n)$ denotes the generic point of $(\mathbb{R}^3)^n$ and $\mathbf{e}_i: h \in \mathbb{R}^3 \mapsto \mathbf{e}_i \cdot h = (0,..., 0, h, 0,..., 0) \in (\mathbb{R}^3)^n$ with h at the ith place.

Let us denote by

$$\mu^{k\varepsilon, n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{V^{k\varepsilon, in}}$$

the empirical measure of this system and by $\pi^{k\varepsilon, n}$ its law, which is a probability measure on $\mathscr{P}(\mathbb{D}([0, T], \mathbb{R}^3))$.

Theorem 4.1. Assume that $Q_0 \in \mathscr{P}_4(\mathbb{R}^3)$. Let $(V_0^i)_{i \geqslant 1}$ be independent Q_0-distributed random variables. Then the sequence $(\pi^{k\varepsilon, n})_{k, \varepsilon, n}$ is uniformly tight for the weak convergence and any limit point charges only probability measures which are solutions of (LMP). Thus any limit point (for the convergence in law) of the sequence $(\mu^{k\varepsilon, n})$ is a solution of (LMP).

Proof. To prove this theorem, we will show

(1) the tightness of $(\pi^{k\varepsilon, n})_{k, \varepsilon, n}$ in $\mathscr{P}(\mathscr{P}(\mathbb{D}([0, T], \mathbb{R}^3)))$,

(2) the identification of the limiting values of $(\pi^{k\varepsilon, n})_{k, \varepsilon, n}$ as solutions of the nonlinear martingale problem (LMP).

One knows (cf. ref. 23) that the tightness of $(\pi^{k\varepsilon, n})_{k, \varepsilon, n}$ is equivalent to the tightness of the laws of the semimartingales $V^{k\varepsilon, 1n}$ belonging to $\mathscr{P}(\mathbb{D}([0, T], \mathbb{R}^3))$. This tightness is due to

$$\sup_{k, \varepsilon, n} E(\sup_{t \leqslant T} |V_t^{k\varepsilon, 1n}|^4) < +\infty. \tag{4.4}$$

This moment condition is obtained by a good use of Burkholder–Davis–Gundy's and Doob's inequalities for (4.2).

Let us now prove that each limiting value of $(\pi^{k\varepsilon, n})$ is a solution of the nonlinear martingale problem (LMP). Consider one of them, denoted by $\pi^\infty \in \mathscr{P}(\mathscr{P}(\mathbb{D}([0, T], \mathbb{R}^3)))$. It is the limiting point of a subsequence we still denote by $(\pi^{k\varepsilon, n})$.

We define, for $\phi \in C_b^1(\mathbb{R}^3)$, $0 \leqslant s_1, ..., s_p \leqslant s < t$, $G \in C_b((\mathbb{R}^3)^p)$, $Q \in \mathscr{P}(\mathbb{D}_T)$ and for X the canonical process on $\mathbb{D}([0, T], \mathbb{R}^3)$, the quantity

$$F(Q) = \left\langle G(X_{s_1}, ..., X_{s_p}) \left(\phi(X_t) - \phi(X_s) - \int_s^t \int_{\mathbb{R}^3} L^\phi(X_u, v_*), Q_u(dv_*)\, du \right), Q \right\rangle. \tag{4.5}$$

Our aim is to prove that $\langle |F|, \pi^\infty \rangle = 0$.

The mapping F is not continuous since the projections are not continuous for the Skorohod topology. However, for any $Q \in \mathscr{P}(\mathbb{D}_T)$, the mapping $X \mapsto X_t$ is Q-almost surely continuous for all t outside of an at most countable set D_Q, and then F is continuous at the point Q if $s, t, s_1, ..., s_p$ are not in D_Q. Here we use the continuity and the boundedness of ϕ, G and also the continuity of $(q, v) \mapsto \int_{\mathbb{R}^3} L^\phi(v, w)\, q(dw)$ on $\mathscr{P}(\mathbb{D}([0, T], \mathbb{R}^3)) \times \mathbb{R}^3$. Thus, if $s, t, s_1, ..., s_p$ are not in D_Q, F is π^∞-a.s. continuous. Then,

$$\langle F^2, \pi^\infty \rangle = \lim_{k, \varepsilon, n} \langle F^2, \pi^{k\varepsilon, n} \rangle$$

But $\langle |F|, \pi^{k\varepsilon, n} \rangle \leqslant \langle |F^{k\varepsilon}|, \pi^{k\varepsilon, n} \rangle + \langle |F - F^{k\varepsilon}|, \pi^{k\varepsilon, n} \rangle$ where

$$F^{k\varepsilon}(Q) = \Big\langle G(X_{s_1}, ..., X_{s_p}) \times \left(\phi(X_t) - \phi(X_s) - \int_s^t \int_{\mathbb{R}^3} K^\phi_{\beta^\varepsilon, k}(X_u, v_*), Q_u(dv_*)\, du \right), Q \Big\rangle \tag{4.6}$$

in which $K^{\phi}_{\beta^{\varepsilon},k}$ is obtained as $K^{\phi}_{\beta^{\varepsilon},\gamma}$ but where $|z|^{\gamma}$ has been replaced by $|z|^{\gamma}\wedge k$. In this case and since $\int_0^{\pi}\beta^{\varepsilon}(\theta)\,d\theta<+\infty$, $K^{\phi}_{\beta^{\varepsilon},k}$ also writes

$$
\begin{aligned}
&K^{\phi}_{\beta^{\varepsilon},k}(v,v_*)\\
&\quad=\int_0^{2\pi}\int_0^{\pi}(\phi(v+a(v,v_*,\theta,\varphi))-\phi(v))\,\psi_k(v-v_*)\,\beta^{\varepsilon}(\theta)\,d\theta\,d\varphi\\
&\quad=\int_0^{2\pi}\int_0^{\pi}\int_0^{kH}(\phi(v+a(v,v_*,\theta,\varphi)\,\mathbf{1}_{\{x\leqslant\psi_k(v-v_*)\}})-\phi(v))\,dx\,\beta^{\varepsilon}(\theta)\,d\theta\,d\varphi.
\end{aligned}
$$

Firstly,

$$
\begin{aligned}
\langle(F^{k\varepsilon})^2,\pi^{k\varepsilon,n}\rangle&=E((F^{k\varepsilon}(\mu^{k\varepsilon,n}))^2)\\
&=E\left(\left(\frac{1}{n}\sum_{i=1}^{n}(M_t^{k\varepsilon,i\phi}-M_s^{k\varepsilon,i\phi})\,G(V_{s_1}^{k\varepsilon,in},\dots,V_{s_p}^{k\varepsilon,in})\right)^2\right)\\
&=\frac{1}{n}\,E(((M_t^{k\varepsilon,1\phi}-M_s^{k\varepsilon,1\phi})\,G(V_{s_1}^{k\varepsilon,1n},\dots,V_{s_p}^{k\varepsilon,1n}))^2)\\
&\quad+\frac{n-1}{n}\,E((M_t^{k\varepsilon,1\phi}-M_s^{k\varepsilon,1\phi})(M_t^{k\varepsilon,2\phi}-M_s^{k\varepsilon,2\phi})\\
&\quad\times G(V_{s_1}^{k\varepsilon,1n},\dots,V_{s_p}^{k\varepsilon,1n})\,G(V_{s_1}^{k\varepsilon,2n},\dots,V_{s_p}^{k\varepsilon,2n}))
\end{aligned}
\tag{4.7}
$$

where $M^{k\varepsilon,i\phi}$ is the martingale defined by

$$
\begin{aligned}
M_t^{k\varepsilon,i\phi}&=\phi(V_t^{k\varepsilon,in})-\phi(V_0^{i})-\frac{1}{n-1}\sum_{j=1}^{n}\int_0^t\int_0^{kH}\int_0^{2\pi}\int_0^{\pi}\\
&\quad(\phi(V_s^{k\varepsilon,in}+a(V_s^{k\varepsilon,in},V_s^{k\varepsilon,jn},\theta,\varphi)\,\mathbf{1}_{\{x\leqslant\psi_k(V_s^{k\varepsilon,in}-V_s^{k\varepsilon,jn})\}})\\
&\quad-\phi(V_s^{k\varepsilon,in}))\,\beta^{\varepsilon}(\theta)\,d\theta\,d\varphi\,dx\,ds
\end{aligned}
$$

and with Doob–Meyer process given by

$$
\begin{aligned}
\langle M^{k\varepsilon,i\phi}\rangle_t&=\frac{1}{n-1}\sum_{j=1}^{n}\int_0^t\int_0^{kH}\int_0^{2\pi}\int_0^{\pi}\\
&\quad(\phi(V_s^{k\varepsilon,in}+a(V_s^{k\varepsilon,in},V_s^{k\varepsilon,jn},\theta,\varphi)\,\mathbf{1}_{\{x\leqslant\psi_k(V_s^{k\varepsilon,in}-V_s^{k\varepsilon,jn})\}})\\
&\quad-\phi(V_s^{k\varepsilon,in}))^2\,\beta^{\varepsilon}(\theta)\,d\theta\,d\varphi\,dx\,ds
\end{aligned}
$$

and for $i \neq j$,

$$\langle M^{k\varepsilon, i\phi}, M^{k\varepsilon, j\phi} \rangle_t$$

$$= \frac{1}{n-1} \int_0^t \int_0^{kH} \int_0^{2\pi} \int_0^{\pi}$$

$$(\phi(V_s^{k\varepsilon, in} + a(V_s^{k\varepsilon, in}, V_s^{k\varepsilon, jn}, \theta, \varphi)\, \mathbf{1}_{\{x \leqslant \psi_k(V_s^{k\varepsilon, in} - V_s^{k\varepsilon, jn})\}} - \phi(V_s^{k\varepsilon, in}))$$

$$\times (\phi(V_s^{k\varepsilon, jn} + a(V_s^{k\varepsilon, jn}, V_s^{k\varepsilon, in}, \theta, \varphi)\, \mathbf{1}_{\{x \leqslant \psi_k(V_s^{k\varepsilon, in} - V_s^{k\varepsilon, jn})\}} - \phi(V_s^{k\varepsilon, jn}))$$

$$\times \beta^{\varepsilon}(\theta)\, d\theta\, d\varphi\, dx\, ds. \tag{4.8}$$

The right terms in (4.7) go to 0 thanks to the expression of the Doob–Meyer process, to the uniform integrability proved in (4.4). Moreover the convergence is uniform on k, ε. Hence

$$\lim_n \langle |F^{k\varepsilon}|, \pi^{k\varepsilon, n} \rangle = 0, \qquad \text{uniformly in } k, \varepsilon.$$

Otherwise, the quantity $\langle |F - F^{k\varepsilon}|, \pi^{k\varepsilon, n} \rangle = E(|F - F^{k\varepsilon}|\, (\mu^{k\varepsilon, n}))$ can be written in an analogous form to the right term of (3.11) replacing Q^{ε} by $\mu^{k\varepsilon, n}$. Its study is thus controled in a similar way than the term E_1 in Section 3.3. Then it converges to 0 uniformly in k and n as ε tends to 0.

Finally, we have proved that

$$\langle |F|, \pi^{\infty} \rangle = 0.$$

Thus, $F(Q)$ is π^{∞}-a.s. equal to 0, for every $s, t, s_1, \ldots, s_p$ outside of the countable set D_Q. It is sufficient to assure that π^{∞}-a.s., Q is a solution of the nonlinear martingale problem (LMP). Let us remark to conclude that each solution Q of the limiting martingale problem is in fact a probability measure on $\mathscr{C}_T$. This remark allows us to deduce immediately the following corollary. ∎

Corollary 4.2. Assume $Q_0 \in \mathscr{P}_4(\mathbb{R}^3)$ and consider a sequence $\mu^{k_r \varepsilon_r, n_r}$ which converges to Q. Then the probability measure-valued process $(\mu_t^{k_r \varepsilon_r, n_r})_{t \geqslant 0}$ converges in probability to the flow $(Q_t)_{t \geqslant 0}$ in the space $\mathbb{D}([0, T], \mathscr{P}(\mathbb{R}^3))$ endowed with the uniform topology.

5. THE MONTE-CARLO ALGORITHM

We deduce from the above study an algorithm associated with the binary mean-field interacting particle system, which enables to observe the transition from the Boltzmann equations to the Landau equation.

At our knowledge, no effective numerical resolution of the Landau equation seen as limit of Boltzmann equations has been obtained by deterministic methods, except in ref. 22 in which a spectral method furnishes a concrete way to study this limit (without numerical resolution). Moreover, according to discussions with numericians, it seems that the deterministic particle methods do not work for the 3D Landau equation. There exist also for this equation some numerical Monte-Carlo algorithms, as Takizuka and Abe[26] and Wang *et al.*,[28] but they are inspired by the diffusion structure of the Landau equation and do not follow the asymptotics of the grazing collisions, and the proofs of convergence are not written.

From now on, the quantities h, γ, k and β^ε defining the cross-section B, the initial distribution Q_0, the terminal time $T>0$ and the size $n\geqslant 2$ of the particle system are fixed. We denote by $B_{k,\varepsilon}(z,\theta)=\psi_k(z)\,\beta^\varepsilon(\theta)$ the corresponding cross-section with cutoff. Because of Theorem 4.1 and Corollary 4.2, we simulate a particle system following (4.3), i.e., the whole path $(V^n_t)_{t\in[0,T]}$ $\in \mathbb{D}([0,T],(\mathbb{R}^3)^n)$.

First of all, we assume that V^n_0 is simulated according to the initial distribution $Q_0^{\otimes n}$. Then, we denote by $0<T_1<\cdots<T_k$ the successive jump times until T of a standard Poisson process with parameter $n\pi kH\,\|\beta^\varepsilon\|_1$.

Before the first collision, the velocities do not change, so that we set $V^n_s=V^n_0$ for all $s<T_1$. Let us describe the first collision. We choose at random a couple (i,j) of particles according a uniform law over $\{(p,m)\in\{1,..., n\}^2; m\neq p\}$. We choose x uniformly on the interval $[0,kH]$, we choose the first angle of collision φ uniformly on $[0,2\pi]$ and we finally choose the collision angle θ following the law $\frac{\beta^\varepsilon(\theta)}{\|\beta^\varepsilon\|_1}\,d\theta$. Then we set

$$
\begin{aligned}
V^{n,i}_{T_1}&=V^{n,i}_0+a(V^{n,i}_0,V^{n,j}_0,\theta,\varphi)\,\mathbf{1}_{\{x\leqslant\psi_k(V^{n,i}_0-V^{n,j}_0)\}}\\
V^{n,j}_{T_1}&=V^{n,j}_0+a(V^{n,j}_0,V^{n,i}_0,\theta,\varphi)\,\mathbf{1}_{\{x\leqslant\psi_k(V^{n,i}_0-V^{n,j}_0)\}}\\
V^{n,p}_{T_1}&=V^{n,p}_0\qquad\text{if}\quad p\neq\{i,j\}
\end{aligned}
$$

Since nothing happens between T_1 and T_2, we set $V^n_s=V^n_{T_1}$ for all $s\in[T_1,T_2[$.

Iterating this method, we simulate $V^n_{T_1},V^n_{T_2},..., V^n_{T_k}$, i.e., the whole path $(V^n_t)_{t\in[0,T]}$, which was our aim.

Notice that this algorithm is very simple and takes a few lines of program and does not require to discretize time. It furthermore conserves momentum and kinetic energy. Let us remark that at least formally, this algorithm can be adapted in a similar way to the Coulombian case, since the soft potential term is cut off for the simulations.

6. NUMERICAL RESULTS

We use the previous Monte-Carlo algorithm to estimate the fourth-order moment of a solution of the Landau equation. By this method, one conserves momentum and kinetic energy, and one follows the asymptotics of grazing collisions.

We consider the cross-section $B_{k,\varepsilon}(z,\theta)=\psi_k(z)\,\beta^\varepsilon(\theta)$ with $\psi_k(z)=|z|\wedge k$ and β^ε satisfying Assumptions (2.3), (3.5) and (3.6).

For each ε, k, we denote by $Q^{k,\varepsilon}$ the solution of the martingale problem with cross-section $B_{k,\varepsilon}$ obtained in Theorem 2.6. We know that for each ε, k, $(Q^{k,\varepsilon})$ is a cluster point, as n tends to infinity, of the empirical measure $\mu^{k,\varepsilon,n}$ associated with a simulable particle system. We also know that $(Q^{k,\varepsilon})_{\varepsilon>0,k\geqslant 0}$ is tight and that any limiting point P is a solution of the martingale problem (LMP) associated with the Landau equation.

At last, we define:

$$m_\gamma^{k,\varepsilon,n}(t)=\int_{\mathbb{R}^3}|v|^4\,\mu_t^{k,\varepsilon,n}(dv);$$

$$m_\gamma^{k,\varepsilon}(t)=\int_{\mathbb{R}^3}|v|^4\,Q_t^{k,\varepsilon}(dv)\qquad\text{and}$$

$$m_\gamma(t)=\int_{\mathbb{R}^3}|v|^4\,P_t(dv).$$

We mention that there is no explicit computation of the fourth-order moment m_t for the Landau equation in our context.

6.1. The "Moderately Soft" Potential Case, $\gamma\in(-1,0]$

We fix $\gamma=-0.8$ and we consider the following asymptotics

$$\beta^\varepsilon(\theta)=\frac{1}{2\pi\varepsilon^3\sin\left(\frac{\theta}{2\varepsilon}\right)^2}\mathbf{1}_{\varepsilon\leqslant|\frac{\theta}{\varepsilon}|\leqslant\pi}$$

These functions satisfy Assumptions (2.3) for any $\varepsilon>0$ and (3.5), (3.6) when ε tends to zero. We notice that $\|\beta^\varepsilon\|_1=\frac{1}{\pi\varepsilon^2}\tan^{-1}(\varepsilon/2)$ and $\Lambda^\varepsilon=\pi\int\beta^\varepsilon(\theta)\sin^2(\frac{\theta}{2})\,d\theta$ converges towards $\Lambda=\pi\ln 2$ as ε tends to 0.

We also consider the initial distribution on $\mathbb{R}^3$, $Q_0(dv)=\mathbf{1}_{[-1/2;1/2]^3}(v)\,dv$.

We first estimate $m_{-0.8}(t)$ at time $t=\frac{1}{2\pi}$. We consider $n=50000$ particles.

First of all, when we consider the mean over 100 simulations of $m_{-0.8}^{k,0.1,50000}(\frac{1}{2\pi})$, we notice that it converges very fastly in k. Hence the error due to the *spatial* cutoff is small:

k	1	4	6	10	50
$m_{-0.8}^{k,0.1,50000}(\frac{1}{2\pi})$	0.09742	0.09873	0.09881	0.09878	0.09875

So we fix $k = 6$ in all what follows.

We now study the convergence of $m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi})$ as ε tends to zero. Taking each time the mean over 100 simulations, we observe in Fig. 1. the convergence of the fourth-order moments for the Boltzmann equation to the one for the Landau equation when ε becomes small.

One can notice that $m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi})$ tends to 0.0988, with a speed of convergence in $|m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi}) - 0.0988| \simeq 0.015 * \varepsilon^2$, when ε tends to zero. Hence, the choice $\varepsilon = 0.1$ seems reasonable to describe the Landau behaviour.

Our algorithm describes precisely the convergence of the Boltzmann equation to the Landau equation. But we take into account all small jumps, then the duration of computation is not optimal. For example, when $\varepsilon = 0.1$ and $k = 6$, there is arround 25.10^6 shocks of particles on the time interval $[0, \frac{1}{2\pi}]$.

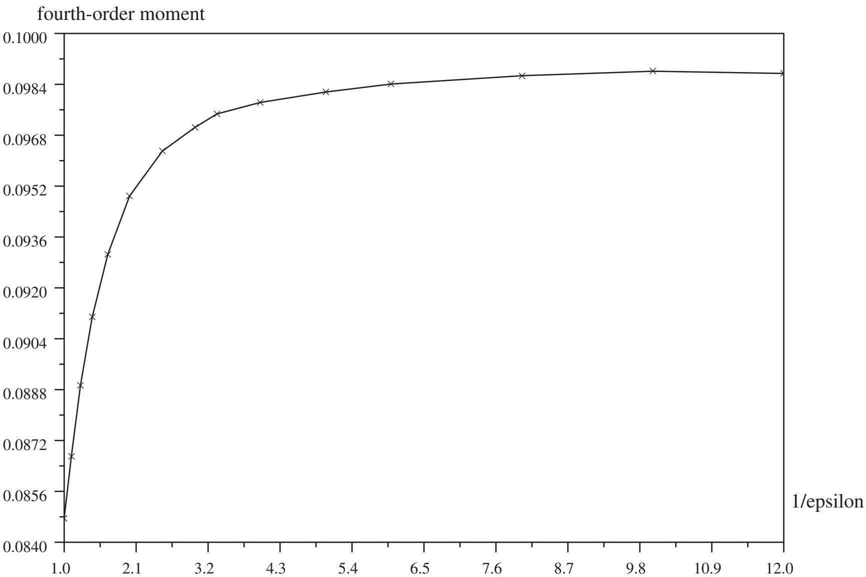

Fig. 1. Evolution in $1/\varepsilon$ of $m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi})$.

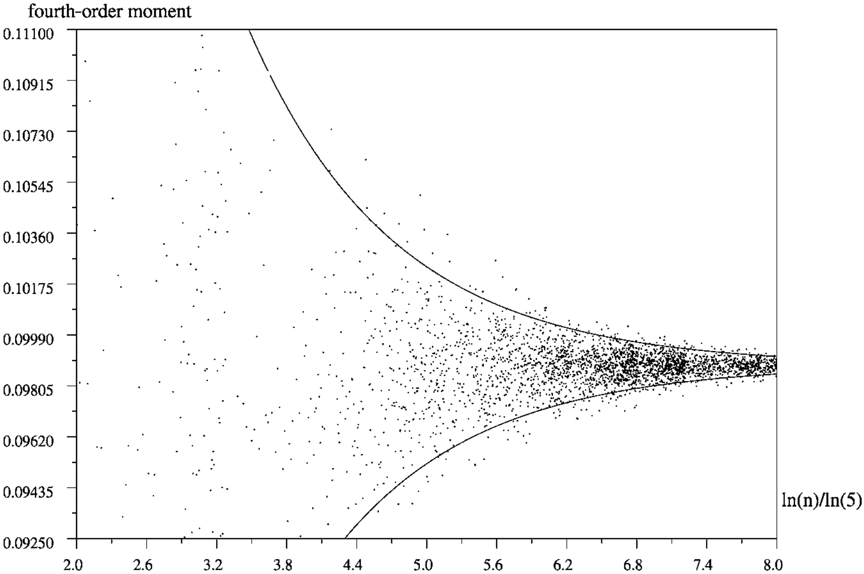

Fig. 2. Evolution of $m_{-0.8}^{6,0.1,n}(\frac{1}{2\pi})$ as $n \to +\infty$. Continuous lines: $0.0988 \pm 0.2/\sqrt{n}$; points: $m_{-0.8}^{6,0.1,n}(\frac{1}{2\pi})$.

Let us now study the speed of convergence of $m_{-0.8}^{6,0.1,n}(\frac{1}{2\pi})$ to $m_{-0.8}^{6,0.1}(\frac{1}{2\pi})$, when n tends to infinity. We obtain the Fig. 2.

The speed of convergence is in $1/\sqrt{n}$. It seems that a central limit theorem holds. (A proof of a similar central limit theorem has been obtained by Fournier and Méléard[10] from 2D Boltzmann equations without cutoff and for Maxwell molecules.)

At last, we observe the evolution in time of the fourth-order moment. (Our method conserves the energy, then the two-order moment is constant in time.) We fix again $k = 6$ and $\varepsilon = 0.1$ and we observe in Fig. 3. the moments of order 4 for some values of $t \in [0, 1]$.

6.2. The Coulombian Case

Our theorical results are satisfied for a potential $\gamma \in (-1, 0]$, but our numerical approach works in the interesting case of Coulomb molecules.

We now consider our algorithm with $\gamma = -3$ and with the same initial condition as in Buet *et al.*[2] We consider $n = 50000$ particles and each value is obtained taking the mean over 100 simulations. We take as initial

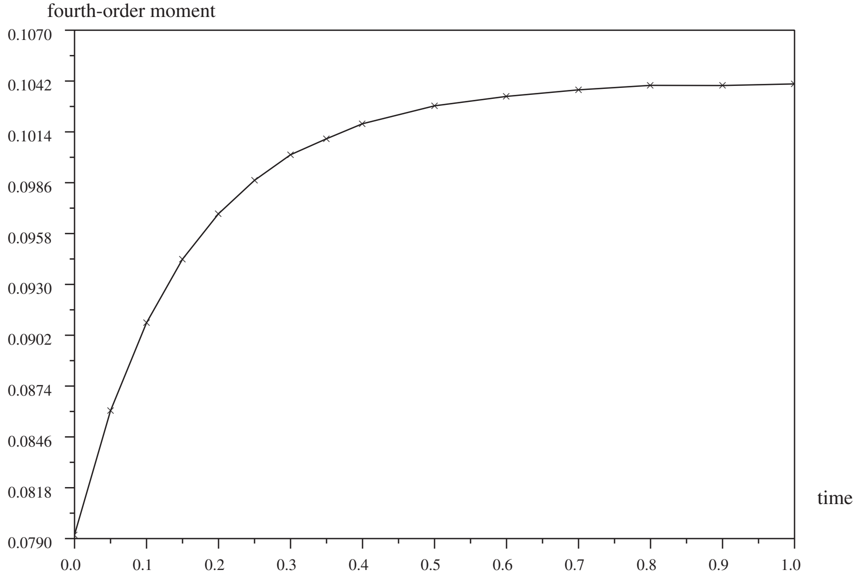

Fig. 3. Evolution in time of $m_{-0.8}^{6,\,0.1,\,50000}(t)$.

condition the measure Q_0 with the following density with respect to the Lebesgue measure:

$$f(0, v) = \tfrac{1}{2}\,(M_{\mathcal{N},\, v_{01},\, v_{th}} + M_{\mathcal{N},\, v_{02},\, v_{th}})$$

where $M_{\mathcal{N},\, u,\, v_{th}}$ is the Maxwellian function on $\mathbb{R}^3$

$$M_{\mathcal{N},\, u,\, v_{th}}(v) = \frac{\mathcal{N}}{(2\pi v_{th}^2)^{3/2}} \exp\left(-\frac{|v-u|^2}{2v_{th}^2}\right)$$

with $\mathcal{N} = 5$, $v_{th} = 0.45$, $v_{01} = (2, 3, 3)$ and $v_{02} = (4, 3, 3)$.

Moreover we take the cross-sections defined in ref. 4, with

$$\beta^{\varepsilon}(\theta) = \frac{1}{|\log \varepsilon|} \frac{\cos(\theta/2)}{\sin^3(\theta/2)}\, \mathbb{I}_{\theta \geqslant \varepsilon}$$

In this situation, Λ^{ε} converges towards $\Lambda = \frac{1}{2}$ as ε tends to 0.

Since the initial datum is not a probability measure, (its mass is equal to 5), we adapt the results obtained by Méléard in ref. 21 and we consider the algorithm with the empirical measure $\mu^{k,\,\varepsilon,\,n} = \frac{5}{n} \sum_{i=1}^{n} \delta_{V^{k\varepsilon,\,in}}$ and the jump times of a standard Poisson process with parameter $\frac{5n\pi k\, \|\beta^{\varepsilon}\|_1}{2}$.

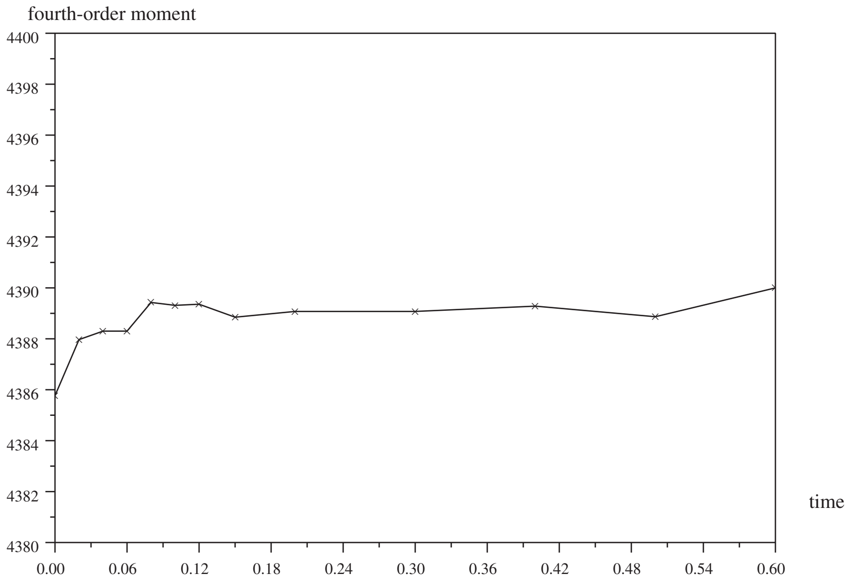

Fig. 4. Evolution in time of $m_{-3}^{6,\,0.2,\,50000}(t)$.

We first estimate the fourth-order moment $m_{-3}(t)$ at time $t = 0.06$.

As for the previous simulations, the algorithm converges very fastly in k. Then we fix again $k = 6$.

We observe that the convergence in ε of the fourth-order moment of the Boltzmann equation to the one of the Landau equation is very fast:

ε	0.9	0.6	0.2	0.1	0.08
$m_{-3}^{6,\,\varepsilon,\,50000}(0.06)$	4389.5	4389.1	4389.9	4388.9	4388.5

The choice of $\varepsilon = 0.2$ seems to be reasonable to describe the Landau moment.

At last, we fix $k = 6$ and $\varepsilon = 0.2$ and we observe in Fig. 4. the evolution in time of the fourth-order moment. We find the same evolution as the one described in ref. 2.

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